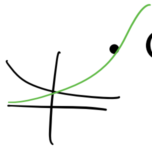
8:00 – 16

3.1 - Linear Models

Various situations are modeled by differential equations.

constant relative growth (or decay) rate

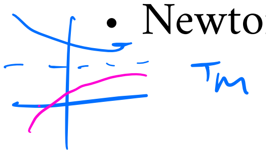


- Growth and decay: $\dfrac{dx}{dt} = kx, \qquad x(t_0) = x_0$

K>0   k<0   initial amount

population, radioactivity, drug concentrations
1st-order chemical reactions   K: growth/decay rate (decimal)

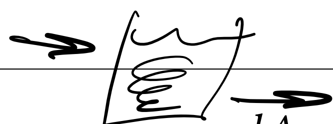
temp



- Newton's law of heating and cooling: $\dfrac{dT}{dt} = k(T - T_m), \qquad T(t_0) = T_0$

time   k<0   initial temperature

T_m is surrounding temperature



- Mixtures: $\dfrac{dA}{dt} = \text{rate in} - \text{rate out}, \qquad A(0) = A_0$
- LR-series electrical circuit: $L\dfrac{di}{dt} + Ri = E(t)$

L: inductance (h) henries   E: impressed voltage volts
R: resistance (Ω) ohms
i: current amps (a)

- RC-series electrical circuit: $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$

C: capicatance farads (f)
q: charge coloumbs (C) $i = \dfrac{dq}{dt}$

Example; A population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?

$\frac{dP}{dt} = kP$ linear (Separable)

$\frac{dP}{dt} - kP = 0 \Rightarrow \mu = e^{-kt}$

initial population

$\frac{d}{dt}(e^{-kt}P) = 0 \Rightarrow P = P_0 e^{kt}$

$P(3) = 400$ $\qquad 400 = P_0 e^{3k}$

$P(10) = 2000$ $\qquad 2000 = P_0 e^{10k}$

$\frac{1}{5} = e^{-7k} \Rightarrow -7k = \ln\frac{1}{5}$

$k = \frac{1}{7}\ln 5$ $\qquad k \approx 0.2299$

$P(t) = P_0 e^{\frac{1}{7}\ln 5\, t}$ OR $P(t) = P_0 e^{0.23t}$

Note: Relative growth rate is $\sim 23\%$

Initial # bacteria is $P(0)$

Sub: $k \rightarrow$ $400 = P_0 e^{3k}$ (arbitrary choice)

$P_0 = 400 e^{-3(0.2299)}$

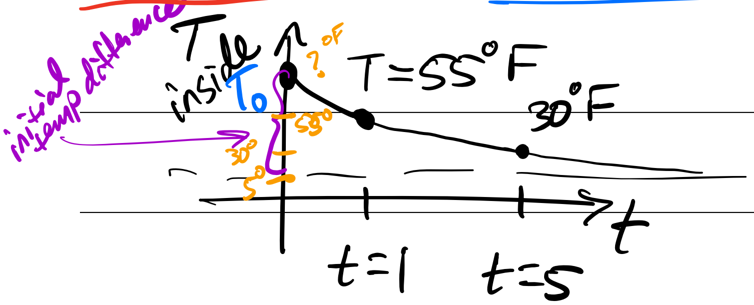
$P_0 \approx 200.6787$

Call it $P_0 \approx 201$ bacteria

Side Note: Time to double: $2P_0 = P_0 e^{kt}$ gives $t = \frac{1}{k}\ln 2$. Decay: half-life is

lambda $\rightarrow \lambda = \frac{1}{k}\ln\frac{1}{2}$

Example: A thermometer is taken from an inside room to the outside, where the air temperature is 5° F. After 1 minute the thermometer reads 55° F, and after 5 minutes it reads 30° F. What is the initial temperature of the inside room?



$$\frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{dT}{dt} - kT = -kT_m$$

$$\mu = e^{-kt} \Rightarrow \frac{d}{dt}(e^{-kt}T) = -e^{-kt}kT_m$$

$$e^{-kt}T = T_m e^{-kt} + C$$

$$T(t) = T_m + C e^{kt}$$

initial temp. difference $T_0 - T_m$

$T(1) = 55°$: $55 = 5 + Ce^{k} \Rightarrow 50 = Ce^{k}$

$T(5) = 30°$: $30 = 5 + Ce^{5k}$ $\quad 25 = Ce^{5k}$

$$2 = e^{-4k} \Rightarrow k = -\tfrac{1}{4}\ln 2$$

$$k \approx -0.1733$$

initial temp. diff.

$$T(t) = 5 + Ce^{-0.18t}$$

$$55 = 5 + Ce^{-0.18} \Rightarrow C = 50e^{0.18}$$

$C \approx 59.8°$ – temp diff

Initial temp is $\sim 64.8°F$.

$A(0)=0$

Example: A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of pounds of salt in the tank at time t.

$$\frac{dA}{dt} = \text{rate salt in} - \text{rate salt out}$$

$$\frac{2\text{ lb}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}} = \frac{10\text{ lb}}{\text{min}}$$

Concentration of salt

$$\frac{A\text{ lb}}{500\text{ gal}} \cdot \frac{5\text{ gal}}{\text{min}} = \frac{A}{100}\frac{\text{lb}}{\text{min}}$$

$$\frac{dA}{dt} = 10 - \frac{1}{100}A \Rightarrow \frac{dA}{dt} + \frac{1}{100}A = 10$$

$$\mu = e^{\frac{1}{100}t} \quad \text{so} \quad \frac{d}{dt}\left(e^{t/100}A\right) = 10e^{t/100}$$

$$e^{t/100}A = 1000e^{t/100} + C$$

$$A = 1000 + Ce^{-t/100}$$

$A(0)=0$ $\qquad C = -1000$

$$A = 1000 - 1000e^{-t/100}$$

OR $\boxed{A(t) = 1000(1-e^{-t/100})}$

amt being subtracted is decreasing

After a long time, 1000 lb salt are in the tank.

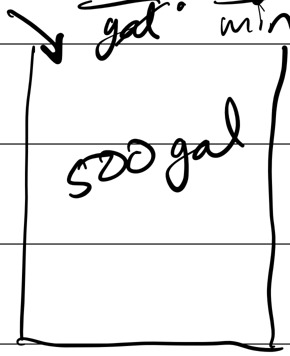
Example: Solve the previous problem under the assumption that the solution is pumped out a a faster rate of 10 gal/min. When is the tank empty?

$\frac{2\text{ lb}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}} = \frac{10\text{ lb}}{\text{min}}$

100 min

5 in − 10 out per min

$= -5$ gal/min



$\frac{A}{\cancel{500-5t}} \cdot \frac{\overset{2}{\cancel{10}}\text{ gal}}{\text{min}}$ 100

5 in − 10 out per min

$$\boxed{\frac{dA}{dt} = \text{rate}_{\text{in}} - \text{rate}_{\text{out}} \qquad \frac{dA}{dt} = 10 - \frac{2A}{100-t}}$$

$$\frac{dA}{dt} + \frac{2}{100-t}A = 10$$

$$\mu = e^{\int \frac{2}{100-t}dt} = e^{-2\ln|100-t|} = \frac{1}{(100-t)^2}$$

$$\frac{d}{dt}\left[\frac{1}{(100-t)^2}A\right] = \frac{10}{(100-t)^2}$$

$$\frac{1}{(100-t)^2}A = \frac{10}{100-t} + C$$

$$-10\int \frac{-1}{(100-t)^2}dt$$

$u = 100-t$

$du = -dt$

$$A(0) = 0 \Rightarrow C = -\frac{1}{10}$$

$$A(t) = 10(100-t) - \frac{1}{10}(100-t)^2$$

OR $\boxed{A(t) = 1000 - 10t - \frac{1}{10}(100-t)^2}$, $0 \le t < 100$

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Example: A 200-volt electromotive force is applied to an RC-series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4$. Determine the charge and current at $t = 0.005$ s. Determine the charge as $t \to \infty$.

$i = \frac{dq}{dt}$ $q(t)$ $i(t)$

$E = 200$, $R = 1000$, $C = 5\times 10^{-6} = \frac{5}{10^6}$

$1000\frac{dq}{dt} + 0.2\times 10^6 q = 200$ $\rightarrow \frac{1}{C} = 5\times 10^6 \rightarrow \frac{10^6}{5}$

$\frac{dq}{dt} + 0.2\times 10^3 q = 0.2$

$\mu = e^{200t} \Rightarrow \frac{d}{dt}(e^{200t}q) = 0.2e^{200t}$

$\frac{0.2}{200} = \frac{0.1}{100} = \frac{1}{1000}$

$e^{200t}q = \frac{1}{1000}e^{200t} + C$

$q(t) = \frac{1}{1000} + Ce^{-200t}$

As $t \to \infty$, $q \to \frac{1}{1000}$ coloumbs

$q'(t)$ $i(t) = -200Ce^{-200t}$, $i(0) = 0.4$

$0.4 = -200C \Rightarrow C = -\frac{0.4}{200} = -\frac{2}{1000}$

$i(t) = 0.4e^{-200t}$

$q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}$

$i(0.005) \approx 0.147$ amps

$q(0.005) \approx 0.000264$ coloumbs